Oct 17, 2022

Week 7

2020A Adv. Cal. II

Another Fubini's theorem

Let $B = [a,b]\times[c,d]\times[e,f]$ and f defined in B.

Recall

$$\iiint_B f \approx \sum_{i,j,k} f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

$$= \sum_{i,j} \left(\sum_k f(x_i^*, y_j^*, z_k^*)\Delta z_k\right)\Delta x_i \Delta y_j$$

$$\approx \sum_{i,j} \int_e^f f(x_i^*, y_j^*, z)\,dz\; \Delta x_i \Delta y_j$$

$$\approx \iint_{[a,b]\times[c,d]} \int_e^f f(x,y,z)\,dz\,dA(x,y)$$

$$= \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx, \text{ which}$$

is the formula we have been using. However, we can put the bracket in a different way.

$$\iiint_B f \approx \sum f(x_i^*, y_j^*, z_k^*)\,\Delta x_i \Delta y_j \Delta z_k$$

$$= \sum_k \left(\sum_{i,j} f(x_i^*, y_j^*, z_k^*)\Delta x_i \Delta y_j\right)\Delta z_k$$

$$\approx \sum_k \iint_{[a,b]\times[c,d]} f(x,y,z_k^*)\,dA(x,y)\,\Delta z_k$$

$$= \int_e^f \iint_{[a,b]\times[c,d]} f(x,y,z)\,dA(x,y)\,dz.$$

When Ω is bounded between two planes $z=e$, $z=f$, let

$$\Omega_z = \{(x,y) : (x,y,z) \in \Omega\}$$

be the cross section of Ω at height z. For $\Omega \subset B$,

$$\iiint_\Omega f = \iiint_B \tilde{f}$$

$$= \int_e^f \iint_{[a,b]\times[c,d]} \tilde{f}(x,y,z)\, dA(x,y)\, dz$$

$$= \int_e^f \iint_{\Omega_z} f(x,y,z)\, dA(x,y)\, dz \qquad (\because \tilde{f}(x,y,z)=0 \text{ for } (x,y) \notin \Omega_z)$$

— ①

In particular, taking $f \equiv 1$ in Ω, we get

$$\text{vol } \Omega = \iiint_\Omega 1\, dV$$

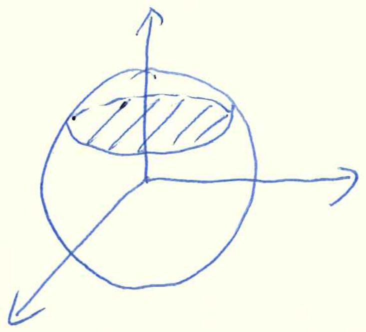
$$= \int_e^f \iint_{\Omega_z} 1\, dA(x,y)\, dz$$

$$= \int_e^f |\Omega_z|\, dz, \quad |\Omega_z| \text{ — area of } \Omega_z.$$

— ②

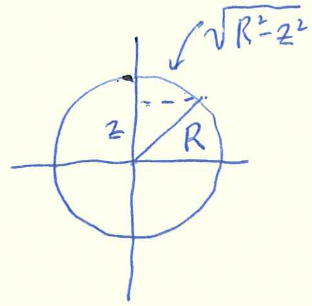
Formulas ① and ② are another forms of Fubini's theorem.

e.g. Find the volume of the ball $x^2+y^2+z^2 \le R^2$ using ②



The cross section at height z is a disk of radius $\sqrt{R^2-z^2}$, as seen from

$$x^2+y^2+z^2 \le R^2$$
$$x^2+y^2 \le R^2-z^2$$



So $|\Omega_z| = \pi\left(\sqrt{R^2-z^2}\right)^2$

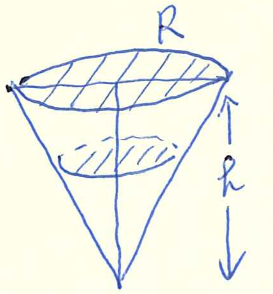
$= \pi(R^2-z^2)$.

vol of the ball $= 2\int_0^R \pi(R^2-z^2)\,dz$

$= 2\pi\left(R^2 z - \frac{z^3}{3}\right)\Big|_0^R$

$= \frac{4\pi}{3}R^3$ #

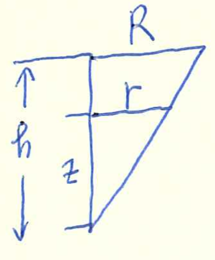
e.g. Find the volume of the circular cone $z = \frac{h}{R}\sqrt{x^2+y^2}$.



$\frac{r}{z} = \frac{R}{h} \Rightarrow r = \frac{R}{h}z$,

the cross section at z is a disk of radius $\frac{R}{h}z$.

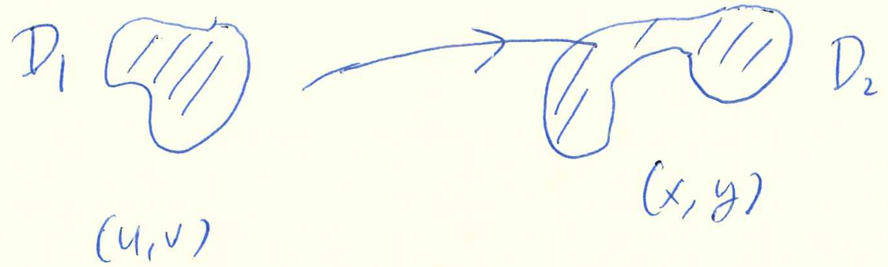
$\therefore |\Omega_z| = \pi\left(\frac{R}{h}z\right)^2$



$$\text{Vol} = \int_0^h \pi\left(\frac{R}{h}z\right)^2 dz = \frac{R^2}{h^2}\pi\frac{h^3}{3} = \frac{1}{3}\pi R^2 h \;\#$$

Change of Variables formula

Let $x = g(u,v)$, $y = h(u,v)$ be a change of variables



Assumptions

1. $(u,v) \mapsto (x,y)$ maps D_1 onto D_2
2. the interior of D_1 is mapped 1-1 to the interior of D_2
3. g and h are continuously differentiable.
4. the inverse map from the interior of D_2 to the interior of D_1 is also continuously diff.

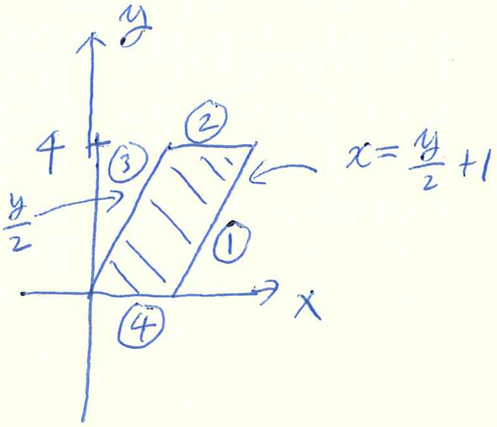
Under ①–④,

$$\iint_{D_2} f = \iint_{D_1} \hat{f}(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dA(u,v).$$

Here $\hat{f}(u,v) = f(g(u,v), h(u,v))$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix} = g_u h_v - g_v h_u .$$

e.g. $\displaystyle\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy$.



Here D_2 is $\quad y/2 \le x \le y/2+1$
$\qquad\qquad\quad 0 \le y \le 4$

Introduce $\quad u = \frac{2x-y}{2}, \ v = \frac{y}{2}$,

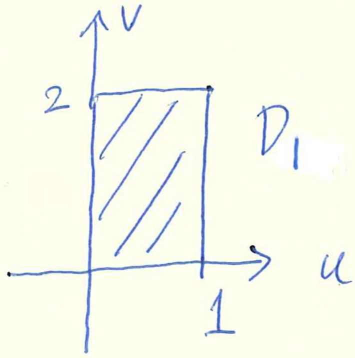
To describe D_1, we see how the sides ①–④ correspond:

① $x = \frac{y}{2}+1, \quad x - \frac{y}{2} = 1, \quad u = 1$

② $y = 4, \quad v = 2$

③ $x = \frac{y}{2}, \quad x - \frac{y}{2} = 0, \quad u = 0$

④ $y = 0, \quad v = 0$



So D_1 is bounded by $u=1, v=2, u=0, v=0$

Now we calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

$$u = \frac{2x-y}{2}, \quad v = \frac{y}{2}$$

$$\Rightarrow u+v = x, \quad y = 2v$$

$\therefore \begin{cases} x = u+v \\ y = 2v \end{cases}$ is the change of variables

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2 > 0$$

Finally,

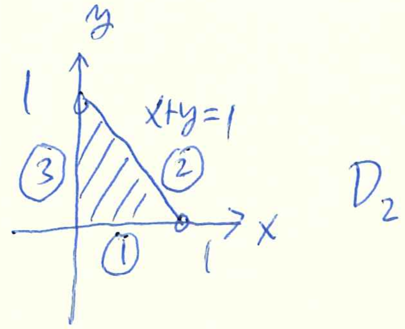
$$\int_0^4 \int_{y/2}^{y/2+1} \frac{2x-y}{2}\,dx\,dy = \iint_{D_2} \frac{2x-y}{2}\,dA(x,y)$$

$$= \iint_{D_2} u \, 2 \, dA(u,v)$$

$$= \int_0^1 \int_0^2 2u \, dv \, du$$

$$= 2 \ \#$$

e.g. $\displaystyle\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy \, dx$



To simplify the integrand, try

$u = x+y, \quad v = y-2x.$

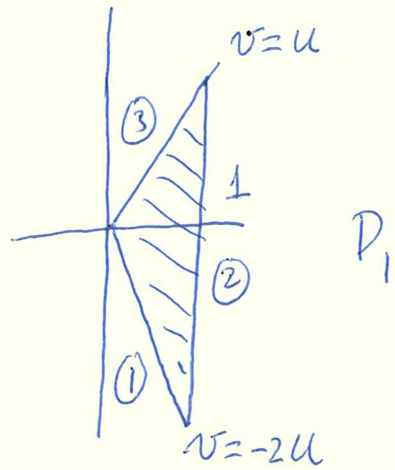
(1) $y=0$, then $u=x, v=-2x$

$\therefore v=-2u$

(2) $x+y=1$, $\quad u=1$

(3) $x=0$, $\quad u=y, v=y$

$\therefore u=v$



Next, solve $u=x+y, v=y-2x$ to get

$$\begin{cases} x = \dfrac{u}{3} - \dfrac{v}{3} \\ y = \dfrac{2u}{3} + \dfrac{v}{3} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Finally,

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2 \, dy \, dx = \iint_{D_2} \sqrt{x+y}\,(y-2x)^2 \, dA(x,y)$$

$$= \iint_{D_1} \sqrt{u}\, v^2 \frac{1}{3}\, dA(u,v)$$

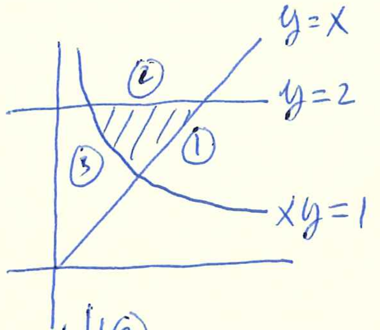
$$= \frac{1}{3} \int_0^1 \int_{-2u}^{u} u^{\frac{1}{2}} v^2 \, dv\, du$$

$\vdots$

$$= \frac{2}{9} \#$$

e.g. $\displaystyle\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

Let $u = \sqrt{xy}$, $v = \sqrt{y/x}$ simplify the integrand.



$D_2 : \ 1/y \le x \le y$

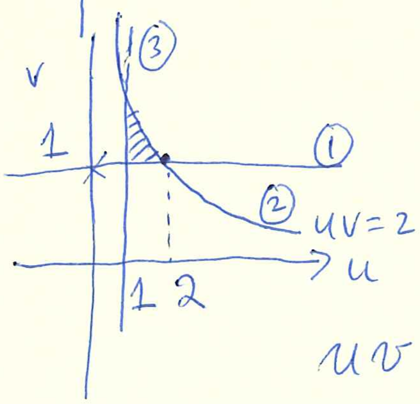
$\qquad 1 \le y \le 2$

① $y = x$, $u = \sqrt{xy} = x$

$\qquad v = 1$

② $y = 2$, $u = \sqrt{2x}$, $v = \sqrt{\frac{2}{x}}$, $uv = 2$

③ $xy = 1$, $u = 1$



$$uv = \sqrt{xy}\sqrt{\frac{y}{x}} = y$$

$$\frac{u}{v} = \frac{\sqrt{xy}}{\sqrt{y/x}} = x$$

$$\therefore \begin{cases} x = u/v \\ y = uv \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}$$

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{\frac{2}{u}} v e^{u} \frac{2u}{v}\, dv\, du$$

$$= \int_1^2 \int_1^{\frac{2}{u}} 2u e^{u}\, dv\, du$$

$$\vdots$$

$$= 2e(e-2) \ \#$$

e.g. Let D_2 be described as

$$g_1(\theta) \le r \le g_2(\theta)$$

$$\theta_1 \le \theta \le \theta_2$$

that is, already in polar form, D_1.

$$\iint_{D_2} f(x,y)\, dA(x,y) = \iint_{D_1} f(r\cos\theta, r\sin\theta) \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| dA(r,\theta).$$

As $x = r\cos\theta$, $y = r\sin\theta$,

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

So, $$\iint_{D_2} f(x,y)\, dA(x,y) = \int_{\theta_1}^{\theta_2} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

the old formula.

(Cont'd)

The change of variables formula in space is

$$\iiint_{\Omega_2} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} \hat{f}(u,v,w) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV(x,y,z)$$

under the setting: $x = g(u,v,w)$, $y = h(u,v,w)$, $z = j(u,v,w)$ maps Ω_1 onto Ω_2, where the interior of Ω_1 is 1-1. Besides, g, h, j are continuously differentiable together with their inverse. Here

$$\hat{f}(u,v,w) = f(g(u,v,w), h(u,v,w), j(u,v,w))$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} g_u & g_v & g_w \\ h_u & h_v & h_w \\ j_u & j_v & j_w \end{vmatrix}$$

e.g. The spherical coordinates

$$x = \rho \sin\varphi \cos\theta = g(\rho, \varphi, \theta)$$
$$y = \rho \sin\varphi \sin\theta = h(\rho, \varphi, \theta)$$
$$z = \rho \cos\varphi = j(\rho, \varphi, \theta)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{vmatrix}$$

$$= \cos\varphi \begin{vmatrix} \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$-(-\rho\sin\varphi) \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta \end{vmatrix}$$

$$+ 0 \times \begin{vmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta \end{vmatrix}$$

$$= \cos\varphi\,(\rho^2\sin\varphi\cos\varphi\cos^2\theta - (-\rho\sin\varphi\sin\theta)\rho\cos\varphi\sin\theta)$$
$$+ \rho\sin\varphi\,(\sin\varphi\cos\theta\,\rho\sin\varphi\cos\theta - (-\rho\sin\varphi\sin\theta)\sin\varphi\sin\theta)$$
$$= \cos\varphi\,(\rho^2\sin\varphi\cos\varphi) + \rho\sin\varphi\,(\rho\sin^2\varphi)$$
$$= \rho^2\sin\varphi\,.$$

So
$$\iiint_{\Omega_2} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} \hat{f}(\rho,\varphi,\theta)\,\rho^2\sin\varphi\,dV(\rho,\varphi,\theta)$$

is recovered.

e.g. Evaluate $\displaystyle\int_0^3\int_0^4\int_{x=y/2}^{x=y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz$

Ω_2 : $\;y/2 \le x \le y/2 + 1,$

$0 \le y \le 4$

$0 \le z \le 3$

Let $u = \frac{2x-y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$.

then $x = u+v$, $y = 2v$, $z = 3w$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

$$\Omega_1 : \begin{array}{l} 0 \le u \le 1, \\ 0 \le v \le 2 \\ 0 \le w \le 1 \end{array}$$

$$\therefore \int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\,dy\,dz$$

$$= \int_0^1 \int_0^2 \int_0^1 (u+w) \frac{1}{6} du\,dv\,dw$$

$$= 12 \ \#$$

In class I described how to get the factor

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

in the formula. Look up the reference listed in the Course Outline for a fuller discussion (if you like).